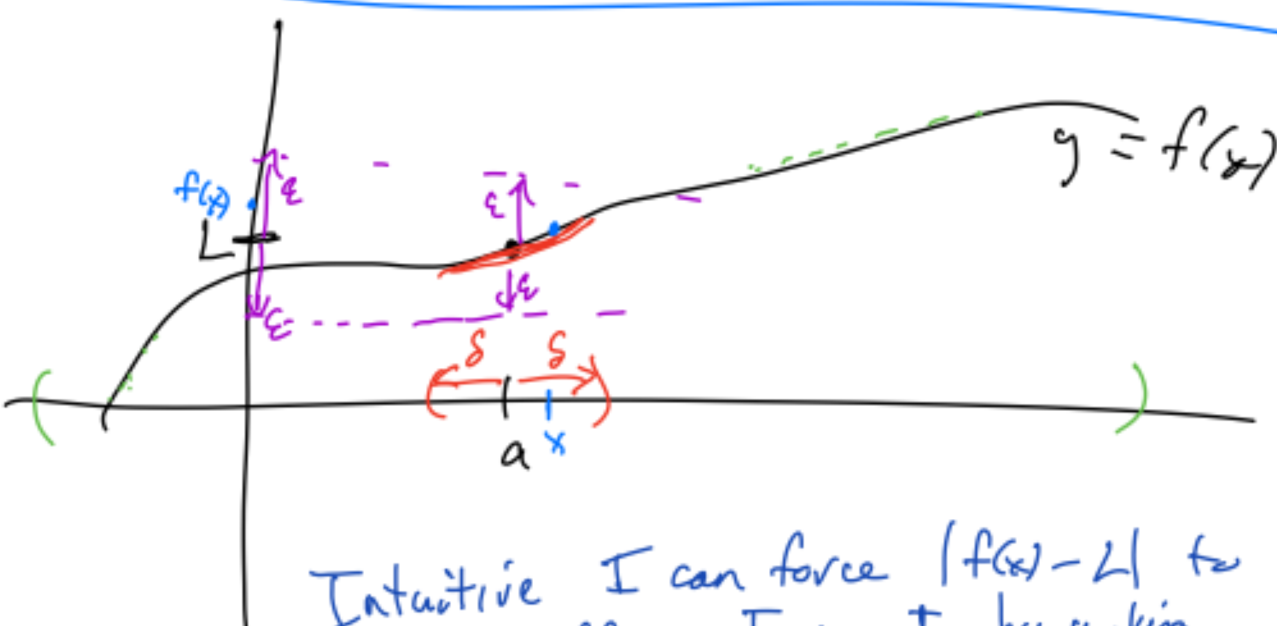
Definition (Real variable version of defn of limit.)

We say $\lim_{x \to a} f(x) = L$ when we mean:

① a is a limit point of the domain of f.

② $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $0 < |x-a| < \delta$ and $x \in \text{domain}(f)$, then $|f(x) - L| < \varepsilon$



Intuitive I can force $|f(x) - L|$ to be as small as I want by making $|x-a|$ really small.

Good news: defn is the same in complex variable, if we use the complex abs. value.

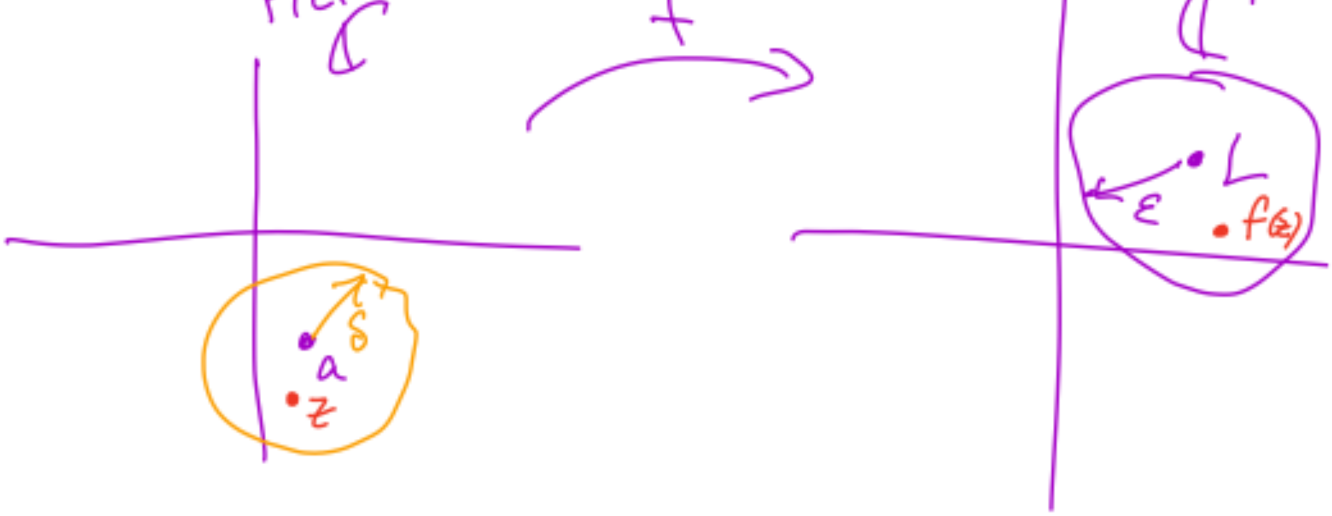
Defn: We say that if f is a complex-valued fcn,

$$\lim_{z \to a} f(z) = L \quad \text{if}$$

① a is a limit point of the domain of f

② $\forall \varepsilon > 0,\ \exists \delta > 0$ such that if $0 < |z-a| < \delta$ and $z \in \text{domain}(f)$, then $|f(z) - L| < \varepsilon$.

Picture.



Definition of Continuous.

If f is a complex valued fcn, we say f is continuous at $a \in \mathbb{C}$

if ① a is in domain(f) $\Longleftrightarrow$ $f(a)$ is defined.

② $\lim_{z \to a} f(z)$ exists $\Longleftrightarrow$ $f(a)$ exists

③ $\lim_{z \to a} f(z) = f(a)$

(Exactly same as "continuous" for real-valued fcn.)

Tips on proving fcn is continuous.

① $z \to 0 \Longleftrightarrow |z| \to 0$

$\underbrace{f(z) \to 0}_{\text{calculus of complex \#s.}} \Longleftrightarrow \underbrace{|f(z)| \to 0}_{\text{calculus of real \#s}}$

Why is this true? $z \to 0 \Leftrightarrow |z - 0| < \varepsilon$

$|z| \to 0 \Leftrightarrow \big||z| - 0\big| < \varepsilon$

$|z - 0| = |z|$, $\big||z| - 0\big| = |z|$

(2) You can convert any limit to the type in (1).

$$\lim_{z \to a} f(z) = L \qquad g(z) = g(w+a)$$

$$\Leftrightarrow \lim_{\underbrace{(z-a)}_{w} \to 0} \left(f(z) - L\right) = 0$$

(3) <u>Squeeze Theorem</u> Real valued limits only:

If $f(x) \le g(x) \le h(x)$ and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then $\lim_{x \to a} g(x)$ exists and $= L$.

Example: find $\lim\limits_{z \to 5} \dfrac{(z-5)^2}{|z|^3 + 7 - 2|z|}$.

$$\frac{(z-5)^2}{|z|^3+7-2|z|} = \frac{w^2}{|w+5|^3+7-2|w+5|}$$

let $w = z-5 \to 0$

$z = w+5$

$$0 \le \left| \frac{w^2}{|w+5|^3+7-2|w+5|} \right| = \frac{|w|^2}{\left| |w+5|^3+7-\underbrace{2|w+5|} \right|}$$

≥ 0

if $|w|<1$

$|w+5| \le |w|+5$

≤ 6

$$\le \frac{|w|^2}{1}$$

$|w+5|^3+7-2|w+5|$ $4 \le 5-|w| \le |w+5|$

$\ge 4^3+7-2(6) \ge 1$

For $|w|<1$,

$$0 \le \left|\frac{w^2}{|w+5|^3+7-2|w+5|}\right| \le |w|^2.$$

Since $\lim_{|w|\to 0} |w|^2 = 0$ from calculus,

by the squeeze thm,

$$\lim_{|w|\to 0} \left|\frac{w^2}{|w+5|^3+7-2|w+5|}\right| = 0$$

$$\therefore \lim_{w\to 0} \frac{w^2}{|w+5|^3+7-2|w+5|} = 0$$

$$\Rightarrow \lim_{z\to 5} \frac{(z-5)^2}{|z|^3+7-2|z|} = 0. \qquad \blacksquare$$

How do we prove a limit does not exist?

Important fact:

• If $\lim_{z\to a} f(z) = L$, then

If $\gamma(t)$ is any continuous curve in $\mathbb{C}$ such that $\gamma(0) = a$, then

$$\lim_{z\to a} f(z) = L = \lim_{t\to 0} f(\gamma(t)) = L$$

$\Longleftrightarrow$ If we can show that on two different paths, α & β, $\lim_{t\to 0} f(\alpha(t)) \neq \lim_{t\to 0} f(\beta(t))$,

$\alpha(0)=\beta(0)=a$ then $\lim_{z\to a} f(z)$ DNE.

Example. Prove that $\lim_{z\to 0} \frac{z}{\operatorname{Re}(z)}$ does not exist.

$H(z)$

Proof: Let $\alpha(t) = t$

$\beta(t) = t(1+i)$

Then $\lim_{t\to 0} \alpha(t) = \lim_{t\to 0} \beta(t) = 0$.

But $\lim_{t\to 0} H(\alpha(t)) = \lim_{t\to 0} \frac{t}{\operatorname{Re}(t)} = \lim_{t\to 0} \frac{t}{t} = \boxed{1}$

and $\lim_{t\to 0} H(\beta(t)) = \lim_{t\to 0} \frac{t(1+i)}{\operatorname{Re}(t(1+i))}$

$$= \lim_{t\to 0} \frac{t(1+i)}{t} = \boxed{1+i}.$$

Since these limits are different, $\lim_{z\to 0} \frac{z}{\operatorname{Re}(z)}$ DNE. $\boxtimes$

When do we know limits exist?

Algebraic Limit Theorem

Suppose $\lim\limits_{z\to a} f(\ )$ & $\lim\limits_{z\to a} g(z)$ exist.

Then

(a) $\lim\limits_{z\to a}\big(f(z) \pm g(z)\big) = \lim\limits_{z\to a}\big(f(z)\big) \pm \lim\limits_{z\to a} g(z)$

(b) $\lim\limits_{z\to a}\big(f(z)g(z)\big) = \Big(\lim\limits_{z\to a} f(z)\Big)\Big(\lim\limits_{z\to a} g(z)\Big).$

(c) $\lim\limits_{z\to a} \overline{f(z)} = \overline{\lim\limits_{z\to a} f(z)}$

(d) $\lim\limits_{z\to a} \underset{\text{Im}}{\mathrm{Re}}\big(f(z)\big) = \underset{\text{Im}}{\mathrm{Re}}\Big(\lim\limits_{z\to a} f(z)\Big)$

(e) $\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}$

if $\lim_{z\to a} g(z) \neq 0$, $g(z) \neq 0$ for z near a.

⋮

- If f is continuous at L & $\lim_{z\to a} g(z) = L$, then

$$\lim_{z\to a} f(g(z)) = f(L).$$

As a consequence:

① Polynomials are continuous.

② Rational fcns are continuous on their domains.

③ Any complex fcn defined by a Taylor series is continuous where the series converges. (open disk)

Limit of sequence.

real-valued

$$\lim_{n\to\infty} z_n = L \iff \lim_{n\to\infty} |z_n - L| = 0$$